

Dynamic Variational Preferences ^(★)

Preferenze Variazionali Dinamiche

EXTENDED ABSTRACT

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Riassunto: Introduciamo e assiomatizziamo le preferenze variazionali dinamiche, la versione intertemporale delle preferenze variazionali che abbiamo assiomatizzato in [10], le quali – a loro volta – generalizzano le preferenze *multiple priors* di Gilboa e Schmeidler [3] e includono sia le preferenze *multiplier* ispirate al controllo stocastico e introdotte in macroeconomia da Hansen e Sargent (si veda [4]), sia le classiche preferenze *media-varianza* di Markovitz e Tobin. Otteniamo inoltre una condizione che rende le preferenze variazionali dinamiche temporalmente coerenti e la loro rappresentazione ricorsiva; ciò garantisce la trattabilità analitica necessaria nelle applicazioni macroeconomiche e finanziarie.

Keywords: ambiguity aversion, model uncertainty, recursive utility, robust control, time consistency.

1. Introduction

In the Multiple Priors (MP) model agents rank acts h using the criterion

$$V(h) \equiv \inf_{p \in C} E^p[u(h)], \quad (1)$$

where C is a closed and convex subset of the set Δ of all probabilities on states. This model has been axiomatized by Gilboa and Schmeidler [3] with the goal of modeling ambiguity averse agents, who exhibit the Ellsberg-type behavior first observed in the seminal paper of Ellsberg [1].

The nonsingleton nature of C reflects the limited information available to agents, which may not be enough to quantify their beliefs with a single probability, and is instead compatible with a nonsingleton set C of probabilities.

On the other hand, the cautious attitude featured by MP agents can also be viewed as the result of the effect that an adversarial influence, which we may call “Nature,” has

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on the realizations of the state. Under this view, Nature chooses a probability p over states with the objective of minimizing agents' utility, conditional on their choice of an act and under the constraint that the probability p has to be chosen in a fixed set C . This interpretation of the MP model provides an intuitive notion of ambiguity aversion, which can be regarded as the agents' diffidence for any lack of precise definition of the uncertainty involved in a choice, something that provides room for the malevolent influence of Nature.⁽¹⁾

In a recent paper, [10], we extended the MP representation by generalizing Nature's constraint. Specifically, in our extension the constraint on Nature is given by a cost $c(p)$ associated with the choice of probability, and agents rank acts according to the criterion:

$$V(h) \equiv \inf_{p \in \Delta} (E^p[u(h)] + c(p)), \quad (2)$$

where c is a closed and convex function on Δ . Preferences represented by (2) are called *variational preferences* (VP), and the function c is their *ambiguity index*. In [10] we axiomatize the representation (2) and we discuss in detail its ambiguity interpretation.

The VP representation generalizes the MP representation, which is the special case where there is an infinite cost for choosing outside the set C , with the cost being constant (and hence, without loss of generality, zero) inside that set. In other words, the cost for Nature in the MP model is given by the indicator function of Convex Analysis of C , defined for all $p \in \Delta$ as

$$\delta_C(p) \equiv \begin{cases} 0 & \text{if } p \in C, \\ \infty & \text{if } p \notin C, \end{cases} \quad (3)$$

and it is immediate to see that

$$\inf_{p \in \Delta} (E^p[u(h)] + \delta_C(p)) = \inf_{p \in C} E^p[u(h)].$$

The notion of ambiguity aversion has found an important application in the last years in the literature, pioneered by Hansen and Sargent (see, e.g., [4] and [5] for details and references), that applies the idea of robust control to agents' choices in macroeconomic models. While the initial definition of robust control was different from that of ambiguity aversion, the intuition is closely related: an agent prefers a robust control if he is not confident that his (probabilistic) model of the uncertainty is correct, and so he wants to avoid the possibility that a small error in the formulation of the stochastic environment produces a large loss. Ambiguity aversion comes up because the agents' information is too limited to be represented by a single probabilistic model.

In the *multiplier preferences* model, the most important choice model used in this macroeconomic literature (see [4]), the constraint on Nature is represented by a cost c

⁽¹⁾ As Hart, Modica, and Schmeidler [6, p. 352] write “In Gilboa and Schmeidler [3] it is shown that preferences ... are represented by functionals of the form $f \mapsto \min_{q \in Q} \sum_s u(f(s)) q(s)$, for some closed convex set $Q \subset \Delta(S)$. So the ambiguity averse decision maker behaves ‘as if’ there were an opponent who could partially influence occurrence of states to his disadvantage (i.e., think of the opponent as choosing $q \in Q$).” Though not yet firmly established, this informed opponent interpretation has found some support in recent experimental findings in the psychological and neuroscience literatures (see [8], [9], [7], and [11]).

based on a reference probability $q \in \Delta$: Nature can deviate away from q , but the larger the deviation, the higher the cost. In particular, this cost is assumed to be proportional to the relative entropy $R(p||q)$ between the chosen probability p and the reference probability q ; that is,

$$c(p) \equiv \theta R(p||q),$$

where $\theta > 0$. Multiplier preferences are, therefore, the special case of variational preferences given by

$$V(h) \equiv \inf_{p \in \Delta} (E^p[u(h)] + \theta R(p||q)),$$

and their analytical tractability is important in deriving optimal policies.

Even though the motivation behind multiplier preferences is similar to that used for MP preferences, formally multiplier preferences are not MP preferences. In fact, in [10] we show that they are an example of divergence preferences, a special class of variational preferences featuring tractable cost functions, but which are not MP preferences. Variational preferences are, therefore, the generalization needed in order to encompass both MP and multiplier preferences, as discussed at length in [10].

In view of applications, however, the static analysis of [10] is insufficient and a dynamic extension is required. This is the purpose of the present work, in which we introduce and axiomatize dynamic variational preferences.

The first observation to make is that, while in a static environment acts are functions from states to consequences, in a dynamic environment they are functions from times and states to consequences. We impose on acts the usual measurability conditions ensuring that agents' choices are consistent with the information they have. As a result, agents' evaluations are conditional to time and state, and they are modelled by a family of (conditional) preferences $\succsim_{t,\omega}$ indexed by time and state pairs (t, ω) . As main results, we provide necessary and sufficient conditions guaranteeing that agents' preferences at time t are represented by the preference functional

$$V_t(h) \equiv \inf_{p \in \Delta} \left(E^p \left[\sum_{\tau \geq t} \beta^{\tau-t} u(h_\tau) \middle| \mathcal{G}_t \right] + c_t(p|\mathcal{G}_t) \right), \quad (4)$$

and we show what restrictions on c_t guarantee time consistency.⁽²⁾ Under time consistency the representation (4) becomes recursive, and so it has the analytical tractability required in applications.

Besides tractability, time consistency has also an intuitive appeal. In fact, suppose that two acts are the same in every contingency up to the present period, and the first is preferred to the second according to the conditional preference in the next period in every state. Then time consistency requires that the first act should be preferred to the second in the present period. Equivalently, think of a plan as a sequence of conditional choices, so that the choice of a plan in the current period includes a plan of choices in all future periods, conditional on all future contingencies. Then, an agent is time consistent if he never formulates a plan of future choices that he wants to revise later in some event that is conceivable today.

⁽²⁾ Here β is a discount factor and $\mathcal{G}_t$ represents the information available to the agents at time t .

2. Setup

Time is discrete and varies over $\mathcal{T} \equiv \{0, 1, \dots, T\}$. In our results we model information as an event tree $\{\mathcal{G}_t\}_{t \in \mathcal{T}}$, given and fixed throughout, which is defined on a finite space Ω . The elements of this tree are partitions $\mathcal{G}_t$ of Ω consisting of non-empty sets, with $\mathcal{G}_0 \equiv \{\Omega\}$, $\mathcal{G}_{t+1}$ finer than $\mathcal{G}_t$ for all $t < T$, and $\mathcal{G}_T \equiv \{\{\omega\} : \omega \in \Omega\}$; in particular, $G_t(\omega)$ is the element of $\mathcal{G}_t$ that contains ω . For non-triviality, we assume $T, |\Omega| \geq 2$.

The acts among which agents choose are here given by consumption processes. Formally, *acts* are X -valued adapted processes of the form $h = (h_0, h_1, \dots, h_T)$, where each $h_t : \Omega \rightarrow X$ is $\mathcal{G}_t$ -measurable and takes values on a convex consumption set X . For example, $X = \mathbb{R}$ or $X = \Delta(\mathbb{R})$, the set of all finitely supported probability measures on $\mathbb{R}$ (in both these cases $\mathbb{R}$ is interpreted as the set of monetary prizes).

Denote by $\mathcal{H}$ the set of all acts; we equivalently write $h_t(\omega)$ or $h(t, \omega)$ to denote consumption at time t if ω obtains (and sometimes $h(t, G)$ to denote consumption at time t if $G \in \mathcal{G}_t$ occurs). For all $\alpha \in [0, 1]$, and all $h, h' \in \mathcal{H}$ we set

$$(\alpha h + (1 - \alpha) h')(t, \omega) \equiv \alpha h(t, \omega) + (1 - \alpha) h'(t, \omega) \quad \forall (t, \omega) \in \mathcal{T} \times \Omega.$$

If the values of an act $y \in \mathcal{H}$ depend only on time but not on state, that is, for every fixed t

$$y(t, \omega) = y(t, \omega') = y_t \quad \forall \omega, \omega' \in \Omega,$$

with a little abuse of notation we write $y = (y_0, y_1, \dots, y_T) \in X^T$. Moreover, if $y_0 = \dots = y_T = x$, the act is called constant and, with another little abuse of notation, we denote it by x .

We close by introducing some notation. If $p \in \Delta(\Omega)$, we denote by $p|_{\mathcal{G}_t}$ its restriction to the algebra $\mathcal{A}(\mathcal{G}_t)$ generated by $\mathcal{G}_t$, and by $p(\cdot | \mathcal{G}_t)$ the conditional probability given $\mathcal{G}_t$.

For all $t \in \mathcal{T}$, $\Delta(\Omega, \mathcal{G}_t)$ denotes the set of all probabilities on $\mathcal{A}(\mathcal{G}_t)$, in particular, $\Delta(\Omega, \mathcal{G}_T) = \Delta(\Omega)$. For each $E \in \mathcal{A}(\mathcal{G}_t)$, we set

$$\begin{aligned} \Delta(E, \mathcal{G}_t) &\equiv \{p \in \Delta(\Omega, \mathcal{G}_t) \mid p(E) = 1\} \\ \Delta^{++}(E, \mathcal{G}_t) &\equiv \left\{ p \in \Delta(\Omega, \mathcal{G}_t) \mid \begin{array}{l} p(G) > 0 \quad \forall G \in \mathcal{G}_t : G \subseteq E \\ p(G) = 0 \quad \forall G \in \mathcal{G}_t : G \not\subseteq E \end{array} \right\}. \end{aligned}$$

3. Axioms

Let the binary relations $\succsim_{t, \omega}$ on $\mathcal{H}$ represent the agent's preferences at any time-state node. Next are stated several properties (axioms) of the preference relation, which will be used in the sequel.

Axiom 1 (Conditional preference—CP) For each $(t, \omega) \in \mathcal{T} \times \Omega$:

- (i) $\succsim_{t, \omega}$ coincides with $\succsim_{t, \omega'}$ if $G_t(\omega) = G_t(\omega')$.
- (ii) If $h(\tau, \omega') = h'(\tau, \omega')$ for all $\tau \geq t$ and $\omega' \in G_t(\omega)$, then $h \sim_{t, \omega} h'$.

(i) says that preferences orderings are “adapted” and allows to write $\succsim_{t, G}$ if $G \in \mathcal{G}_t$. (ii) states that at time t in event G only “continuation acts” matter for choice.

Axiom 2 (Variational preferences—VP) For each $(t, \omega) \in \mathcal{T} \times \Omega$:

- (i) $\succsim_{t,\omega}$ is complete and transitive.
- (ii) For all $h, h' \in \mathcal{H}$ and $y, y' \in X^T$, and for all $\alpha \in (0, 1)$, if $\alpha h + (1 - \alpha)y \succsim_{t,\omega} \alpha h' + (1 - \alpha)y'$ then $\alpha h + (1 - \alpha)y' \succsim_{t,\omega} \alpha h' + (1 - \alpha)y$.
- (iii) For all $h, h', h'' \in \mathcal{H}$, the sets $\{\alpha \in [0, 1] : \alpha h + (1 - \alpha)h' \succsim_{t,\omega} h''\}$ and $\{\alpha \in [0, 1] : h'' \succsim_{t,\omega} \alpha h + (1 - \alpha)h'\}$ are closed.
- (iv) For all
 $h, h' \in \mathcal{H}$, if $(h_0(\omega'), h_1(\omega'), \dots, h_T(\omega')) \succsim_{t,\omega} (h'_0(\omega'), h'_1(\omega'), \dots, h'_T(\omega'))$ for all $\omega' \in \Omega$, then $h \succsim_{t,\omega} h'$.
- (v) For all $h, h' \in \mathcal{H}$, if $h \sim_{t,\omega} h'$, then $\alpha h + (1 - \alpha)h' \succsim_{t,\omega} h$ for all $\alpha \in (0, 1)$.
- (vi) There exist $x \succ_{t,\omega} x'$ in X such that for all $\alpha \in (0, 1)$ there is $x'' \in X$ satisfying either $x' \succ_{t,\omega} \alpha x'' + (1 - \alpha)x$ or $\alpha x'' + (1 - \alpha)x' \succ_{t,\omega} x$.

The requirement here is that at every time-state node the agent has (unbounded) variational preferences; see Maccheroni, Marinacci, and Rustichini [10] for a discussion of (i)-(vi).

Axiom 3 (Risk preference—RP) For any $y \in X^T$ and all $x, x', x'', x''' \in X$, if

$$(y_{-\{\tau, \tau+1\}}, x, x') \succsim_{t,\omega} (y_{-\{\tau, \tau+1\}}, x'', x''')$$

holds for some $(t, \omega) \in \mathcal{T} \times \Omega$ and some $\tau \geq t$, then it holds for all $(t, \omega) \in \mathcal{T} \times \Omega$ and all $\tau \geq t$.⁽³⁾

This is a standard stationarity axiom.

Axiom 4 (Dynamic consistency—DC) For each $(t, \omega) \in \mathcal{T} \times \Omega$ with $t < T$, and all $h, h' \in \mathcal{H}$, if $h_\tau = h'_\tau$ for all $\tau \leq t$ and $h \succsim_{t+1, \omega'} h'$ for all $\omega' \in \Omega$, then $h \succsim_{t,\omega} h'$.

As Epstein and Schneider [2, p. 6] observe “According to the hypothesis, h and h' are identical for times up to t , while h is ranked (weakly) better in every state at $t + 1$. ‘Therefore’, it should be ranked better also at (t, ω) .”

A state $\omega'' \in \Omega$ is $\succsim_{t,\omega}$ -null if

$$h(\omega') = h'(\omega') \text{ for all } \omega' \neq \omega'' \text{ implies } h \sim_{t,\omega} h'.$$

Axiom 5 (Full support—FS) No state in Ω is $\succsim_{0,\Omega}$ -null.

4. Results

We first extend to the current dynamic setting the notion of ambiguity index c we used in the static setting of [10]. A *dynamic ambiguity index* is a family $\{c_t\}_{t \in \mathcal{T}}$ of functions $c_t : \Omega \times \Delta(\Omega) \rightarrow [0, \infty]$ such that for all $t \in \mathcal{T}$:

- (i) $c_t(\cdot, p) : \Omega \rightarrow [0, \infty]$ is $\mathcal{G}_t$ -measurable for all $p \in \Delta(\Omega)$,

⁽³⁾ Notation: $(y_{-\{\tau, \tau+1\}}, x, x') \equiv (y_0, \dots, y_{\tau-1}, x, x', y_{\tau+2}, \dots, y_T)$ if $\tau < T$ and $(y_0, \dots, y_{T-1}, x)$ otherwise.

- (ii) $c_t(\omega, \cdot) : \Delta(\Omega) \rightarrow [0, \infty]$ is grounded,⁽⁴⁾ closed and convex, with $\text{dom } c_t(\omega, \cdot) \subseteq \Delta(G_t(\omega))$ and $\text{dom } c_t(\omega, \cdot) \cap \Delta^{++}(G_t(\omega)) \neq \emptyset$, for all $\omega \in \Omega$.

In our first result we characterize a dynamic version of variational preferences that do not necessarily satisfy dynamic consistency. Notice that in (5) we consider $\Delta^{++}(\Omega)$ in order to have well defined conditional probabilities $p_{G_t(\omega)}$.

Proposition 1 *The following statements are equivalent:*

- (a) $\{\succsim_{t,\omega}\}$ satisfy CP, VP, RP, and for each $(t, \omega) \in \mathcal{T} \times \Omega$ no state in $G_t(\omega)$ is $\succsim_{t,\omega}$ -null.
(b) There exist a scalar $\beta > 0$, an unbounded affine function $u : X \rightarrow \mathbb{R}$, and a dynamic ambiguity index $\{c_t\}$ such that, for each $(t, \omega) \in \mathcal{T} \times \Omega$, $\succsim_{t,\omega}$ is represented by

$$V_t(\omega, h) \equiv \inf_{p \in \Delta^{++}(\Omega)} \left(\int \sum_{\tau \geq t} \beta^{\tau-t} u(h_\tau) dp_{G_t(\omega)} + c_t(\omega, p_{G_t(\omega)}) \right) \quad \forall h \in \mathcal{H}. \quad (5)$$

Moreover, $(\beta', u', \{c'_t\})$ represents $\{\succsim_{t,\omega}\}$ in the sense of (5) if and only if $\beta' = \beta$, $u' = au + b$ for some $a > 0$ and $b \in \mathbb{R}$, and $\{c'_t\} = \{ac_t\}$.

As a result, for all $t \in \mathcal{T}$ and all $h \in \mathcal{H}$, the preference functional $V_t(\cdot, h)$ is a $\mathcal{G}_t$ -measurable random variable

$$V_t(h) = \inf_{p \in \Delta^{++}(\Omega)} \left(\mathbb{E}^p \left(\sum_{\tau \geq t} \beta^{\tau-t} u(h_\tau) \mid \mathcal{G}_t \right) + c_t(p \mid \mathcal{G}_t) \right).$$

We call *dynamic variational preferences* the (families of) preferences satisfying CP, VP, RP, and such that no state in $G_t(\omega)$ is $\succsim_{t,\omega}$ -null. It is natural to wonder what restriction on the dynamic ambiguity index would characterize the dynamic variational preferences that satisfy dynamic consistency. This condition is given in the next theorem, which is our main result.

Theorem 2 *The following statements are equivalent:*

- (a) $\{\succsim_{t,\omega}\}$ satisfy CP, VP, RP, FS, and DC.
(b) There exist a scalar $\beta > 0$, an unbounded affine function $u : X \rightarrow \mathbb{R}$, and a dynamic ambiguity index $\{c_t\}$ such that, for each $(t, \omega) \in \mathcal{T} \times \Omega$, $\succsim_{t,\omega}$ is represented by

$$V_t(\omega, h) \equiv \inf_{p \in \Delta^{++}(\Omega)} \left(\int \sum_{\tau \geq t} \beta^{\tau-t} u(h_\tau) dp_{G_t(\omega)} + c_t(\omega, p_{G_t(\omega)}) \right) \quad \forall h \in \mathcal{H}, \quad (6)$$

⁽⁴⁾ That is, $\min_{p \in \Delta(\Omega)} c_t(\omega, p) = 0$.

and

$$c_t(\omega, q) = \beta \sum_{\substack{G \in \mathcal{G}_{t+1} \\ q(G) > 0}} q(G) c_{t+1}(G, q_G) + \min_{\{p \in \Delta(G_t(\omega)) : p|_{\mathcal{G}_{t+1}} = q|_{\mathcal{G}_{t+1}}\}} c_t(\omega, p), \quad (7)$$

for all $q \in \Delta(G_t(\omega))$ and all $t < T$.

Moreover, $(\beta', u', \{c'_t\})$ represents $\{\succsim_{t,\omega}\}$ in the sense of (6) if and only if $\beta' = \beta$, $u' = au + b$ for some $a > 0$ and $b \in \mathbb{R}$, and $\{c'_t\} = \{ac_t\}$.

Therefore, dynamic variational preferences satisfy dynamic consistency if and only if their dynamic ambiguity index has the recursive structure (7).

In turn, (7) delivers the recursive representation

$$V_t(\omega, h) = u(h_t(\omega)) + \min_{r \in \Delta(\Omega, \mathcal{G}_{t+1})} \left(\beta \int V_{t+1}(h) dr + \gamma_t(\omega, r) \right) \quad (8)$$

of the agent's preference functional V_t , where

$$\gamma_t(\omega, r) = \min_{\{p \in \Delta(G_t(\omega)) : p|_{\mathcal{G}_{t+1}} = r\}} c_t(\omega, p) \quad \forall r \in \Delta(\Omega, \mathcal{G}_{t+1}). \quad (9)$$

In view of all this, we call *recursive variational preferences* the dynamic variational preferences satisfying dynamic consistency, and we call *recursive ambiguity indexes* their dynamic ambiguity indexes, that is, the dynamic indexes satisfying the no-gain condition (7) for some $\beta > 0$.

The recursive formula (7) has a transparent interpretation under the game against Nature interpretation of our setting, in which $\{c_t\}$ is a dynamic cost for Nature. In fact, (7) suggests that the cost for Nature of choosing q at time t in state ω can be decomposed as the sum of: the discounted expected cost of choosing q 's conditionals at time $t + 1$, plus the cost $\gamma_t(\omega, q|_{\mathcal{G}_{t+1}})$ of inducing $q|_{\mathcal{G}_{t+1}}$ as one-period-ahead marginal. By (7) and (8), both Nature's costs and agent's preferences are recursive.

Our results extend Epstein and Schneider [2]'s characterization of dynamic MP preferences, and condition (7) generalizes their rectangularity condition (coinciding with it when cost functions are indicators of Convex Analysis).

A main advantage of the recursive structure of condition (7) is that it permits the construction by backward induction of recursive ambiguity indexes, and so of recursive variational preferences via (8) and (9).

The next result provides the key ingredient for the desired backward induction construction

Proposition 3 *Let $\{c_t\}$ be a dynamic ambiguity index. For all $t < T$ and $\omega \in \Omega$, set*

$$\gamma_t(\omega, r) \equiv \min_{\{p \in \Delta(G_t(\omega)) : p|_{\mathcal{G}_{t+1}} = r\}} c_t(\omega, p) \quad \forall r \in \Delta(\Omega, \mathcal{G}_{t+1}).$$

The family $\{\gamma_t\}_{t < T}$ of functions $\gamma_t : \Omega \times \Delta(\Omega, \mathcal{G}_{t+1}) \rightarrow [0, \infty]$ is such that for all $t < T$:

- (i) $\gamma_t(\cdot, r) : \Omega \rightarrow [0, \infty]$ is $\mathcal{G}_t$ -measurable for all $r \in \Delta(\Omega, \mathcal{G}_{t+1})$.
- (ii) $\gamma_t(\omega, \cdot) : \Delta(\Omega, \mathcal{G}_{t+1}) \rightarrow [0, \infty]$ is grounded, closed and convex, with $\text{dom } \gamma_t(\omega, \cdot) \subseteq \Delta(G_t(\omega), \mathcal{G}_{t+1})$ and $\text{dom } \gamma_t(\omega, \cdot) \cap \Delta^{++}(G_t(\omega), \mathcal{G}_{t+1}) \neq \emptyset$, for all $\omega \in \Omega$.

The index $\gamma_t(\omega, r)$ can be interpreted as the cost for Nature of inducing r as one-period-ahead marginal, as suggested by (8) and (9). Since the properties of $\gamma_t(\omega, \cdot)$ on $\Delta(\Omega, \mathcal{G}_{t+1})$ are analogous to those of a static (or dynamic) ambiguity index on the set of the agent's subjective beliefs, we call *one-period-ahead ambiguity index* a family $\{\gamma_t\}_{t < T}$ of functions that satisfies conditions (i) and (ii) of Proposition 3.

Next we characterize recursive ambiguity indexes by means of one-period-ahead ones, thus giving the desired backward induction construction of recursive ambiguity indexes. Here δ_C is the indicator function defined in (3) and, given $\omega \in \Omega$, d_ω is the Dirac probability assigning mass 1 to ω .

Theorem 4 *Let $\{c_t\}_{t \in T}$ be a family of functions from $\Omega \times \Delta(\Omega)$ to $[0, \infty]$. The following statements are equivalent:*

- (a) $\{c_t\}$ is a recursive ambiguity index.
- (b) There exist $\beta > 0$ and a one-period-ahead ambiguity index $\{\gamma_t\}$ such that, for all $\omega \in \Omega$,

$$c_T(\omega, \cdot) = \delta_{\{d_\omega\}}, \text{ and for all } t < T$$

$$c_t(\omega, q) = \begin{cases} \beta \sum_{\substack{G \in \mathcal{G}_{t+1} \\ q(G) > 0}} q(G) c_{t+1}(G, q_G) + \gamma_t(\omega, q_{|\mathcal{G}_{t+1}}) & \forall q \in \Delta(G_t(\omega)) \\ \infty & \forall q \in \Delta(\Omega) \setminus \Delta(G_t(\omega)). \end{cases}$$

In this case, $\{\gamma_t\}$ is unique and satisfies (9).

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